Advances in Applied Mathematics 48 (2012) 99–105

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Noncommutative biorthogonal polynomials

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ARTICLE INFO

Article history:
Received 27 September 2010
Accepted 25 May 2011
Available online 22 June 2011

MSC:
15A15
42C05

Keywords:
Biorthogonal polynomials
Quasideterminants
Recurrence relations

ABSTRACT

We define and study biorthogonal sequences of polynomials over noncommutative rings, generalizing previous treatments of biorthogonal polynomials over commutative rings and of orthogonal polynomials over noncommutative rings. We extend known recurrence relations for specific cases of biorthogonal polynomials and prove a general version of Favard's theorem.

1. Introduction

The theory of orthogonal polynomials is well established and has many applications. For any sequence $\{S_i\}$ of elements of a commutative ring R, we can define a biadditive function $\langle\cdot,\cdot\rangle : R[x] \times R[x] \to R$ by $\langle ax^i, bx^j\rangle = abS_{i+j}$ for $a, b \in R$ and define a sequence of polynomials $\{p_n\}$ by

$$p_n = \begin{vmatrix} S_n & \cdots & S_{2n-1} & x^n \\ \vdots & \ddots & \vdots & \vdots \\ S_0 & \cdots & S_{n-1} & 1 \end{vmatrix}.$$

Then $\langle p_n, p_m\rangle = 0$ if and only if $n \neq m$, i.e. the sequence $\{p_n\}$ is orthogonal. The S_i are called the moments of $\{p_n\}$. For a more detailed introduction see either [3] or [13], Chihara's and Szego's classic texts on the subject. The idea of orthogonal polynomials and this method of generating them has been generalized in two ways to achieve new types of polynomials: noncommutative orthogonal polynomials and biorthogonal polynomials.

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0196-8858/$ – see front matter
doi:10.1016/j.aam.2011.05.003

The theory of orthogonal polynomials has been extended to cover rings of noncommutative operators, in particular matrices. The study of orthogonal matrix polynomials started with Krein, for instance in [10] and [11], and developed significantly in the mid 1990s; for example in [4,5,12,6] by Duran, Van Assche and others. In [8], Gelfand, Krob, Lascoux, Leclerc, Retakh and Thibon, using the notion of quasideterminants introduced in [9] (see also [7]), extended the theory to general noncommutative rings by setting p_n equal to the quasideterminant of a similar matrix. The paper also shows that the 3-term recurrence relation, which is well-known for commutative orthogonal polynomials, still holds in this case.

Second, orthogonal polynomials have been generalized in several ways to biorthogonal polynomials. See [2] for more details on these generalizations. One such extension is considered in [1] by Bertola, Gekhtman and Szmigielski. A family of biorthogonal polynomials is defined to be two sequences of real polynomials $\{p_n(x)\}$ and $\{q_m(y)\}$ with the property that $\iint p_n(x)q_m(y)K(x,y)\,d\alpha(x)\,d\beta(y) = 0$ when $n \neq m$ for particular K, α and β. In this paper, it is shown that these polynomials can be represented as determinants of matrices whose entries are bimoments and, for a specific $K(x,y)$, a 4-term recurrence relation is obtained.

Here, we define biorthogonal polynomials over a noncommutative ring. We bring together the two different generalizations described above to present a completely algebraic definition of noncommutative biorthogonal polynomials. For our purposes, a biorthogonal family consists of two sequences of polynomials $\{p_n(x)\}$ and $\{q_n(y)\}$, over a division ring R, along with a function $\langle\cdot,\cdot\rangle$: $R[x] \times R[y] \to R$ so that $\langle p_n(x), q_m(y)\rangle = 0$ for all $n \neq m$. Using this definition, we obtain recurrence relations for some types of biorthogonal polynomials and thus generalize the 4-term recurrence relations of [1]. We conclude with a broad extension of Favard's theorem.

2. Set-up and definitions

Let R be a division ring with center C. We will view $R[x]$ as an R-C bimodule of R and $R[y]$ as a C-R bimodule of R. That is, elements of $R[x]$ will be of the form $\sum a_i x^i$ and elements of $R[y]$ will be of the form $\sum y^j b_j$ so that $xc = cx$ and $yc = cy$ for all $c \in C$. Let $\langle\cdot,\cdot\rangle : R[x] \times R[y] \to R$ so that

$$\left\langle \sum a_i x^i, \sum y^j b_j \right\rangle = \sum a_i \langle x^i, y^j \rangle b_j.$$

A system of polynomials $\{p_n\}, \{q_n\}_{n\in\mathbb{N}}$ is *biorthogonal* with respect to $\langle\cdot,\cdot\rangle$ if $\langle p_n(x), q_m(y)\rangle = 0$ for all $n \neq m$.

Let $I_{a,b} = \langle x^a, y^b\rangle$. The set $I = \{I_{a,b}\}_{a,b\in\mathbb{Z}_{\geqslant 0}}$ is called the set of *bimoments* for $\langle\cdot,\cdot\rangle$. The bimoments completely define the function $\langle\cdot,\cdot\rangle$ so we will say that a set of polynomials is biorthogonal with respect to I. In keeping with the notation of [1], we will let I be the matrix of bimoments and write Id for the identity matrix. Note in these cases, and below, all matrices and vectors are infinite, with rows and columns indexed by $\mathbb{Z}_{\geqslant 0}$.

We extend $\langle\cdot,\cdot\rangle$ to $R[x]^n \times R[y]$ and to $R[x] \times R[y]^n$ in the following way:

$$\text{if } B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in R[x]^n \quad \text{and} \quad g \in R[y], \quad \text{then } \langle B, g\rangle = \begin{bmatrix} \langle b_1, g\rangle \\ \vdots \\ \langle b_n, g\rangle \end{bmatrix}.$$

Similarly,

$$\text{if } f \in R[x] \quad \text{and} \quad D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in R[y]^n, \quad \text{then } \langle f, D\rangle = \begin{bmatrix} \langle f, d_1\rangle \\ \vdots \\ \langle f, d_n\rangle \end{bmatrix}.$$

If $C \in Mat_{r\times n}(R)$, $B \in R[x]^n$ and $g \in R[y]$, then $\langle CB, g\rangle = C\langle B, g\rangle$.

For an $(n+1) \times (n+1)$ matrix A, let $A^{i,j}$ denote the $n \times n$ matrix formed by removing the i-th row and j-th column.

Then (cf. [9]) the i, j*-quasideterminant of* A $|A|_{i,j}$ is

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i,1} & \cdots & \boxed{a_{i,j}} & \cdots & a_{i,n+1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n+1,1} & \cdots & a_{n+1,j} & \cdots & a_{n+1,n+1} \end{vmatrix}$$

$$= a_{i,j} - [\, a_{i,1} \quad \cdots \quad a_{i,j-1} \quad a_{i,j+1} \quad \cdots \quad a_{i,n+1} \,] \cdot \left(A^{i,j}\right)^{-1} \cdot \begin{bmatrix} a_{1,j} \\ \cdots \\ a_{i-1,j} \\ a_{i+1,j} \\ \cdots \\ a_{n+1,j} \end{bmatrix}.$$

Note that, after suitably permuting rows and columns, this the is Schur complement of a block decomposition of A. The quasideterminant $|A|_{i,j}$ exists if and only if $A^{i,j}$ is invertible.

3. Constructing biorthogonal polynomials using bimoments

Throughout, we will assume that the set of bimoments is generic in the sense that all quasideterminants considered exist and are invertible. This is our only restriction on the set of bimoments.

Theorem. *Let* $\{I_{a,b} \mid a, b \in \mathbb{Z}_{\geqslant 0}\} \subseteq R$*. For all* $n \in \mathbb{N}$*, define*

$$p_n(x) = |I|_{1,n+1} = \begin{vmatrix} I_{n,0} & \cdots & I_{n,n-1} & \boxed{x^n} \\ \vdots & \ddots & \vdots & \vdots \\ I_{1,0} & \cdots & I_{1,n-1} & x \\ I_{0,0} & \cdots & I_{0,n-1} & 1 \end{vmatrix}$$

and

$$q_n(y) = \begin{vmatrix} 1 & y & \cdots & \boxed{y^n} \\ I_{n-1,0} & I_{n-1,1} & \cdots & I_{n-1,n} \\ \vdots & \vdots & \ddots & \vdots \\ I_{0,0} & I_{0,1} & \cdots & I_{0,n} \end{vmatrix}.$$

Then $\{p_n\}, \{q_n\}$ *is a (monic) biorthogonal system of polynomials with respect to the set of bimoments* $\{I_{a,b}\}$*.*

To prove this Theorem we need the following Lemma:

Lemma. *Let* $n \in \mathbb{Z}_{\geqslant 0}$ *and* p_n, q_n *be as defined as in the proposition. Then* $\langle x^i, q_n\rangle = \langle p_n, y^i\rangle = 0$ *for all* $0 \leqslant i \leqslant n-1$.

Proof of Lemma. Let $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n-1$. We see that

$$\langle p_n, y^i \rangle = \left\langle x^n - [\, I_{n,0} \quad \cdots \quad I_{n,n-1} \,] \cdot \left(I^{1,n+1}\right)^{-1} \cdot \begin{bmatrix} x^{n-1} \\ \vdots \\ 1 \end{bmatrix}, y^i \right\rangle$$

$$= I_{n,i} - [\, I_{n,0} \quad \cdots \quad I_{n,n-1} \,] \cdot \left(I^{1,n+1}\right)^{-1} \cdot \begin{bmatrix} I_{n-1,i} \\ \vdots \\ I_{0,i} \end{bmatrix}.$$

Applying the definition of quasideterminant, we see that this is

$$\begin{vmatrix} I_{n,0} & \cdots & I_{n,n-1} & \boxed{I_{n,i}} \\ \vdots & \ddots & \vdots & \vdots \\ I_{0,0} & \cdots & I_{0,n-1} & I_{0,i} \end{vmatrix}.$$

Thus, since $0 \leqslant i \leqslant n-1$, $\langle p_n, y^i \rangle$ is the quasideterminant of a matrix whose n-th column is equal to its $(i+1)$-st column and hence is 0 (cf. [7], Prop. 1.4.6).

Similarly,

$$\langle x^i, q_n \rangle = \begin{vmatrix} I_{i,0} & \cdots & \boxed{I_{i,n}} \\ I_{n-1,0} & \cdots & I_{n-1,n} \\ \vdots & \ddots & \vdots \\ I_{0,0} & \cdots & I_{0,n} \end{vmatrix}.$$

Thus, since $0 \leqslant i \leqslant n-1$, the top row will be equal to the $(n-i+1)$-th row, again making the quasideterminant 0 (cf. [7], Prop. 1.4.6). $\square$

Proof of Theorem. Let $n, m \in \mathbb{N}$ so that $n \neq m$. Suppose $n < m$. Now $p_n(x) = \sum_{k=0}^{n} a_k x^k$ for some $a_0, \ldots, a_n \in R$. Thus $\langle p_n, q_m \rangle = \sum_{k=0}^{n} a_k \langle x^k, q_m \rangle$. For all $0 \leqslant k \leqslant n$, $k < m$ so by the lemma, $\langle x^k, q_m \rangle = 0$. Thus $\langle p_n, q_m \rangle = 0$. The case for $n > m$ is similar. $\square$

Remark. We note here that we can recover the construction of orthogonal polynomials in [8] from the construction above. Let R be the free associative algebra on generators $S_0, S_1, \ldots$ with $S_{a+b} = I_{a,b}$ for all $a, b \in \mathbb{N}$. Following the notation of [8] let $*$ be the anti-automorphism so $(S_k)^* = S_k$ and $(\sum c_i x^i)^* = \sum (c_i)^* x^i$. A little examination shows that $q_n = p_n^*$. Thus $\langle p_n, q_m \rangle = \langle p_n, p_m \rangle_*$, i.e. the collection $\{p_n\}$ is orthogonal with respect to the (very similar) inner product $\langle \cdot, \cdot \rangle_*$ where $\langle \sum c_i x^i, \sum d_j y^j \rangle_* = \sum c_i S_{i+j} (d_j)^*$.

4. Banded matrices

For $i, j \in \mathbb{Z}_{\geqslant 0}$, let $E_{i,j}$ denote the matrix with rows and columns indexed by $\mathbb{Z}_{\geqslant 0}$ so that the (i, j) entry is 1 and all other entries are 0. Let $a \leqslant 0$ and $b \geqslant 0$. $M_{[a,b]}$ is defined to be $span\{E_{i,j}: a \leqslant i - j \leqslant b\}$. We will refer to these matrices as "banded". For example, the set of diagonal matrices is $M_{[0,0]}$. Let $X \in M_{[a,b]}$ and $Y \in M_{[c,d]}$.

Lemma. *$X + Y \in M_{[\min(a,c),\max(b,d)]}$ and $XY \in M_{[a+c,b+d]}$.*

Proof. The proof that $X+Y \in M_{[\min(a,c),\max(b,d)]}$ is trivial. Suppose $[XY]_{u,v} \neq 0$. Then $[X]_{u,w} \neq 0$ and $[Y]_{w,v} \neq 0$ for some w. This implies $a \leqslant w-u \leqslant b$ and $c \leqslant v-w \leqslant d$. Adding these equations shows that $a+c \leqslant v-u \leqslant b+d$. Thus $XY \in M_{[a+c,b+d]}$. $\square$

5. Recurrence relations

In the commutative case, [1] obtains a 4 term recurrence relation when $I_{a+1,b} + I_{a,b+1} = \alpha_a \beta_b$. This means there is a formula for p_{n+1} in terms of p_n, p_{n-1}, and p_{n-2} and a similar formula for q_{n+1}. K is called the kernel of a system of biorthogonal polynomials if $\langle a(x), b(y)\rangle = \iint a(x)b(y)K(x,y)\,dx\,dy$. The condition above corresponds to what the authors of this paper called the "Cauchy kernel": $K(x,y) = \frac{1}{x+y}$. Below, we achieve similar, but longer, recurrences that correspond to kernels of the form $\frac{1}{f(x)+g(y)}$ where f and g are polynomials.

For all $n \in \mathbb{N}$, let

$$p_n = \begin{vmatrix} I_{n,0} & \cdots & \boxed{I_{n,n}} \\ \vdots & \ddots & \vdots \\ I_{0,0} & \cdots & I_{0,n} \end{vmatrix}^{-1} \cdot \begin{vmatrix} I_{n,0} & \cdots & I_{n,n-1} & \boxed{x^n} \\ \vdots & \ddots & \vdots & \vdots \\ I_{0,0} & \cdots & I_{0,n-1} & 1 \end{vmatrix}$$

and let

$$q_n = \begin{vmatrix} 1 & \cdots & \boxed{y^n} \\ I_{n-1,0} & \cdots & I_{n-1,n} \\ \vdots & \ddots & \vdots \\ I_{0,0} & \cdots & I_{0,n} \end{vmatrix}.$$

These are scalar multiples of the polynomials constructed in Section 2. Therefore they are biorthogonal. A quick check will show that we also have that $\langle p_n, q_n\rangle = 1$ for all $n \in \mathbb{N}$. Thus this system of polynomials is *biorthonormal.*

Theorem. *Let $\{p_k\}$, $\{q_k\}$ be any biorthonormal polynomials with bimoments I. Suppose there exist polynomials over the center of R, $f(x) = \sum_{i=0}^n a_i x^i$ and $g(y) = \sum_{j=0}^m y^j b_j$ so that $\sum_{i=0}^n a_i I_{r+i,s} + \sum_{j=0}^m I_{r,s+j} b_j = \alpha_r \beta_s$ for all $r, s \in \mathbb{N}$. Then there exist $n+m+2$ term recurrence relations for p_i and q_i. That is, we can express p_{i+1} in terms of $p_i, \ldots, p_{i-n-m-2}$ and q_{i+1} in terms of $q_i, \ldots, q_{i-n,m-2}$. The recurrences we achieve for p_{i+1} and q_{i+1} have polynomial coefficients for p_i, p_{i-1}, q_i, and q_{i-1} and scalar coefficients for all other terms.*

Proof. Let

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Let $p(x)$ and $q(y)$ be column vectors with entries p_k and q_k respectively. Note that for each $k \in \mathbb{Z}_{\geqslant 0}$, p_k and q_k are polynomials of degree k so for each so the products $p_k f(x)$ and $g(y) q_k$ can be written as a linear combination of $p_{n+k}, \ldots, p_1, p_0$ and $q_{m+k}, \ldots, q_1, q_0$ respectively.

Let X and Y be the infinite scalar matrices so that $p(x) f(x) = X p(x)$ and $g(y) q^T(y) = q^T(y) Y^T$. Since $\langle p(x), q^T(y)\rangle = Id$, we know that $\langle p(x) f(x), q^T(y)\rangle = X$ and $\langle p(x), g(y) q^T(y)\rangle = Y^T$.

Suppose $p_k(x) = \sum_{i=0}^{k} c_i x^i$ and $q_l(y) = \sum_{i=0}^{l} y^i d_i$. Let $\pi_k = \sum_{i=0}^{k} c_i \alpha_i$ and $\eta_l = \sum_{i=0}^{l} \beta_i d_i$,

$$\begin{aligned}(X + Y^T)_{k,l} &= \langle p_k(x) f(x), q_l(y)\rangle + \langle p_k(x), g(y) q_l(y)\rangle \\ &= \sum_{i,j} c_i \langle f(x) x^i, y^j\rangle d_j + \sum_{i,j} c_i \langle x^i, y^j g(y)\rangle d_j \\ &= \sum_{i,j} c_i \alpha_i \beta_j d_j = \pi_k \eta_l.\end{aligned}$$

If π and η are vectors with entries π_n and η_n respectively, then $X + Y^T = \pi \eta^T = D_\pi(\underline{1})(\underline{1}^T) D_\eta$ where D_π and D_η are diagonal matrices with (i,i) entries π_i and η_i, respectively.

Let $A = (\Lambda - Id) D_\pi^{-1} X$ and $B^T = Y^T D_\eta^{-1}(\Lambda^T - Id)$. Since $\underline{1}$ is a null vector of $\Lambda - Id$, $(\Lambda - Id) D_\pi^{-1}(X + Y^T) = 0$ and $(X + Y^T) D_\eta^{-1}(\Lambda^T - Id) = 0$. Then $A = -(\Lambda - Id) D_\pi^{-1} Y^T$ and $B^T = -X D_\eta^{-1}(\Lambda^T - Id)$.

We claim that A and B are banded matrices. Note that $X \in M_{[-\infty,n]}$ since $X_{i,j} = \langle p_i(x) f(x), q_j(y)\rangle = 0$ if $i + n < j$ (because the degree $p_i * f(x)$ is less than the degree of q_j) and that $Y^T \in M_{[-m,\infty]}$ since $Y^T_{i,j} = \langle p_i(x), g(y) q_j(y)\rangle = 0$ if $i > m + j$. Note also that $(\Lambda - I) \in M_{[0,1]}$.

Applying the results we obtained for banded matrices, we see that $A = (\Lambda - Id) D_\pi^{-1} X \in M_{[-\infty,n+1]}$ and that $A = -(\Lambda - Id) D_\pi^{-1} Y^T \in M_{[-m,\infty]}$. Thus $A \in M_{[-m,n+1]}$. Similarly, $B^T \in M_{[-\infty,m+1]}$ and $B^T \in M_{[-n,\infty]}$ so $B^T \in M_{[-n,m+1]}$ and $B \in M_{[-m-1,n]}$.

Recall that $p(x) f(x) = X p(x)$ and $g(y) q^T(y) = q^T(y) Y^T$. Then $(\Lambda - Id) D_\pi^{-1} p(x) f(x) = (\Lambda - Id) D_\pi^{-1} X p(x) = A p(x)$ and $g(y) q^T(y) D_\eta^{-1}(\Lambda^T - Id) = q^T(y) Y^T D_\eta^{-1}(\Lambda^T - Id) = q^T(y) B^T$.

Thus examining the $(k-1)$-th row of these equations gives the following $n+m+2$ term recurrence relations, as desired:

$$\left(\pi_k^{-1} p_k - \pi_{k-1}^{-1} p_{k-1}\right) f(x) = \sum_{i=k-m}^{k+n+1} A_{k-1,i} p_i,$$

$$g(y)\left(\eta_k^{-1} q_k - \eta_{k-1}^{-1} q_{k-1}\right) = \sum_{i=k-n}^{k+m+1} B_{k-1,i} q_i. \qquad \square$$

6. Biorthogonal analogue of Favard's theorem

Favard's theorem states that if $\{p_n(x)\}$ is a sequence of polynomials which obeys the usual 3-term recurrence relation then there exists an inner product for which these polynomials are orthogonal. Here we show that any two sequences of polynomials are biorthogonal with respect to some function, for which we construct the bimoments. It is important to note that no recurrence relation is required here.

Theorem. *Let $\{p_n\}$, $\{q_n\}$ be any set of polynomials over any division ring R so that p_n and q_n are of degree n for all $n \in \mathbb{N}$. For any $\{c_k\}_{k \in \mathbb{Z}_{\geqslant 0}}$ in R, there exists a unique set of bimoments for which $\{p_n\}$, $\{q_n\}$ is a biorthogonal system of polynomials and $\langle p_k, q_k\rangle = c_k$.*

Proof. It is equivalent to show that there is a set of bimoments so that for all $a, b \in \mathbb{N}$, the following conditions hold:

(1) If $a < b$ then $\langle x^a, q_b(y)\rangle = 0$.
(2) If $a > b$ then $\langle p_a(x), y^b\rangle = 0$.
(3) If $a = b$, then $\langle p_a(x), q_b(y)\rangle = c_a$.

We will define $I_{a,b}$ inductively on $a+b$. It is pivotal to note that the equations $\langle x^a, q_b(y)\rangle = 0$, $\langle p_a(x), y^b\rangle = 0$, and $\langle p_a(x), q_b(y)\rangle = c_a$ do not involve bimoments of the form $I_{i,j}$ where $i+j > a+b$. Recall that $p_0, q_0 \in R$. Let $I_{0,0} = p_0^{-1} c_0 q_0^{-1}$. Then $\langle p_0, q_0\rangle = p_0 I_{0,0} q_0 = 1$ as desired. □

Let $n \geqslant 1$ and suppose for all a, b such that $a+b < n$, we have defined $I_{a,b}$ to satisfy the previous conditions. For each $0 \leqslant i \leqslant n$ define $I_{i,n-i}$ as follows:

Case 1. If $i < n-i$ then the equation $\langle x^i, q_{n-i}\rangle = 0$ is a linear equation whose variables (the bimoments) have all been defined except for $I_{i,n-i}$ due to the order in which the $I_{a,b}$'s are defined. Therefore there is a unique solution which we must define $I_{i,n-i}$ to be.

Case 2. Similarly, if $i > n-i$, the equation $\langle p_i, y^{n-i}\rangle = 0$ has only one unknown and thus has a unique solution which we define $I_{i,n-i}$ to be.

Case 3. If $i = n-i$ then, again, the equation $\langle p_i, q_{n-i}\rangle = c_i$ has one unknown and we define $I_{i,n-i}$ to be the unique solution to this linear equation.

At each step we satisfy all the necessary conditions and have no choice so the bimoments constructed are the unique set for which $\{p_n\}, \{q_n\}$ is a biorthogonal system with $\langle p_k, q_k\rangle = c_k$.

Acknowledgments

Special thanks to Professor Robert Wilson for overseeing and guiding this research, as well as all assistance in creating this paper. This paper represents results obtained at a Research Experience for Undergraduates program at Rutgers University and was funded by NSF-0603745.

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